Name (Last, First): ______________________________

Student ID: ______________________________

Circle your section:

201	Shin	8am	71 Evans	212	Lim	1pm	3105 Etcheverry
202	Cho	8am	75 Evans	213	Tanzer	2pm	35 Evans
203	Shin	9am	105 Latimer	214	Moody	2pm	81 Evans
204	Cho	9am	254 Sutardja Dai	215	Tanzer	3pm	206 Wheeler
205	Zhou	10am	254 Sutardja Dai	216	Moody	3pm	61 Evans
206	Theerakarn	10am	179 Stanley	217	Lim	8am	310 Hearst
207	Theerakarn	11am	179 Stanley	218	Moody	5pm	71 Evans
208	Zhou	11am	254 Sutardja Dai	219	Lee	5pm	3111 Etcheverry
209	Wong	12pm	3 Evans	220	Williams	12pm	289 Cory
210	Tabrizian	12pm	9 Evans	221	Williams	3pm	140 Barrows
211	Wong	1pm	254 Sutardja Dai	222	Williams	2pm	220 Wheeler

If none of the above, please explain: ______________________________

This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points, of which you must complete 5. **Choose one problem not to be graded by crossing it out in the box below.** If you forget to cross out a problem, we will roll a die to choose one for you.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total Possible	50	

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1) Decide if the following statements are ALWAYS TRUE **(T)** or SOMETIMES FALSE **(F)**. You do not need to justify your answers. (Correct answers receive 2 points, incorrect answers -2 points, blank answers 0 points.)

a) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent vectors in $\mathbb{R}^6$, then $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_4$ are linearly independent vectors.

(T) If $a(\mathbf{v}_1 + \mathbf{v}_2) + b(\mathbf{v}_3 - \mathbf{v}_4) = 0$ then $a\mathbf{v}_1 + a\mathbf{v}_2 + b\mathbf{v}_3 - b\mathbf{v}_4 = 0$ so $a = b = 0$.

b) The following linear system is inconsistent

$$\begin{array}{rcrcrcrcr} -2x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 10 \\ x_1 & - & 2x_2 & + & 3x_3 & - & 4x_4 & = & -5 \end{array}$$

(F) The system can be written as an augmented matrix and put in reduced row echelon form

$$\left[\begin{array}{cccccc} -2 & 4 & -6 & 8 & | & 10 \\ 1 & -2 & 3 & -4 & | & -5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccccc} 1 & -2 & 3 & -4 & | & -5 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right]$$

There is no row of the form $\left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & | & 1 \end{array}\right]$.

c) If A is a 3×2 matrix and B is a 2×3 matrix, then the rank of the 3×3 matrix AB must be less than or equal to 2.

(T) Since B is 2×3, we have $rank(B) \leq 2$, and so $\dim Nul(B) \geq 1$. Hence $\dim Nul(AB) \geq 1$, and so $rank(AB) \leq 2$.

d) If two $m \times n$ matrices A and B have the same reduced row echelon form, then they have the same column spaces.

(F) Counterexample:

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

e)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix} = 24$$

(T) Row reduce to an upper triangular matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

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2) Circle all of the answers that satisfy the questions below. It is possible that any number of the answers (including none) satisfy the questions. (Complete solutions receive 2 points, partial solutions 1 points, but any incorrect circled answer leads to 0 points.)

a) Let A be an $m \times n$ matrix. Which of the following is equal to m? Solution: v).

i) $rank(A)$

ii) $\dim Col(A) + \dim Nul(A)$

iii) $rank(A^T)$

iv) $\dim Col(A^T) - \dim Nul(A^T)$

v) $\dim Col(A^T) + \dim Nul(A^T)$

b) Which of the following matrices is in reduced row echelon form? Solution: $iii), v)$.

$$i)\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \quad ii)\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad iii)\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad iv)\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad v)\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c) Which of the following conditions insures an $m \times n$ matrix A is invertible? Solution: $iv), v)$.

i) $m = n$.

ii) There exists an $n \times m$ matrix B such that $AB = I_m$.

iii) The row echelon form of A has the same number of pivot rows as pivot columns.

iv) $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every $\mathbf{b}$.

v) A is injective and surjective.

d) Which of the following $T : \mathbb{R}^2 \to \mathbb{R}$ is a linear transformation? Solution: $ii), iv), v)$.

i) $T(x, y) = x + y + 1$

ii) $T(x, y) = x - 2y$

iii) $T(x, y) = x^2 + y^2 - (x + y)^2$

iv) $T(x, y) = 6(x + 1) + 2(y - 3)$

v) $T(x, y) = 0$

e) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ has 2-dimensional range and we know

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Which of the following is a possible value of $T(\mathbf{e}_2)$? Solution: i), iii), iv), v).

$$i)\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \quad ii)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad iii)\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \quad iv)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad v)\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

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3) Consider the matrix

a) (5 points) Find bases for the column space and null space of

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Row reduce to find:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Nul(A)$ basis:

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$Col(A)$ basis:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

b) (5 points) For what values of c is the vector

$$\mathbf{v} = \begin{bmatrix} c \\ 2c \\ c^2 \end{bmatrix}$$

in the column space of A?

Solve system:

$$a \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} c \\ 2c \\ c^2 \end{bmatrix}$$

Can do by row reduction, or observe that must have $b = c$, $a = 2c$ and so $2c - c = c^2$. Thus need $c = 0$ or 1.

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4) (10 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies the following:

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad T(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find the standard matrix of T.

We seek the 2×3 matrix with columns $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$.

We have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1/2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1/2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1/2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1/2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus we have

$$T(\mathbf{e}_1) = T((1/2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1/2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = (1/2)T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) + (1/2)T(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$$

$$= (1/2)T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1/2)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T((1/2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1/2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = (1/2)T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) + (-1/2)T(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$$

$$= (1/2)T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1/2)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

And so the matrix we seek is

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 0 & 0 \end{bmatrix}$$

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5) Decide if each of the following matrices is invertible, and either find its inverse or justify why it is not invertible.

a) (5 points)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row reduce to find inverse:

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 2 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & -1 \\ 0 & 1 & 0 & | & -2 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

b) (5 points)

$$B = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -4 & 4 & 2 & 2 \\ -2 & -4 & -4 & -2 \\ 1 & 2 & -2 & 1 \end{bmatrix}$$

Not invertible: column 2 is twice column 4 so $\det B = 0$.

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6) (10 points) Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{R}^n$ and that A is an $m \times n$ matrix. Prove that if $A\mathbf{v}_1, \ldots, A\mathbf{v}_k$ are linearly independent in $\mathbb{R}^m$, then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Suppose $a_1\mathbf{v}_1 + \cdots a_k\mathbf{v}_k = \mathbf{0}$. We must show $a_1 = \cdots = a_k = 0$.

Apply A to find $A(a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) = \mathbf{0}$, and so $A(a_1\mathbf{v}_1) + \cdots + A(a_k\mathbf{v}_k) = \mathbf{0}$, and so $a_1A\mathbf{v}_1 + \cdots + a_kA\mathbf{v}_k = \mathbf{0}$.

Since $A\mathbf{v}_1, \ldots, A\mathbf{v}_k$ are linearly independent, we have $a_1 = \cdots = a_k = 0$.